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Uncertain buckling: its past, present and future

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Abstract

An authoritative review on stochastic buckling of structures was written by Amazigo some quarter century ago. The present review summarizes some of the developments which took place since then. A brief overview of the effect of uncertainty in the initial geometric imperfections, elastic moduli, applied forces, and thickness variation is given. For the benefit of the thoughtful reader, the review is of critical nature. © 2000 Published by Elsevier Science Ltd.

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1. Introduction

The general theory of buckling and postbuckling behavior of elastic structures and its principal essence – the imperfection sensitivity theory – was worked out by Koiter (1945, 1963a,b, 1967, 1970). Further contributions were provided by Budiansky and Hutchinson (1964), Stein (1968), Elishakoff and Arbocz (1985) and other investigators. For bibliography, the reader may consult, for example, the review articles by Budiansky and Hutchinson (1966, 1979), Hutchinson and Koiter (1970), Stein (1972), Budiansky (1974), Koiter (1976), Arbocz (1991, 1997), and Knight and Starnes (1998).

There are many other investigations dealing with the schism that exists between the theoretical analyses and the experimental results. Most unfortunately, experimental results "misbehave" and do not match the theoretical predictions. In these circumstances, it was not unnatural to look for uncertainty as a responsible factor for the scatter in the experimental results. One conceptually understands that no two identical shells exist even when produced by the same manufacturing procedure. Motivated by this idea, the investigators could ascribe the scatter in buckling loads to the scatter in the initial imperfections.

The next step was the identification of uncertainty with randomness and recourse to the probabilistic methods. We find the first indications of these thoughts in the paper by Hoff (1949).

This idea was pursued by Bolotin (1962), apparently independently. He postulated, in brief, that the buckling load λ of a structure can be expressed as a deterministic function of a finite number of parameters $\overline{\xi}_i$, representing the initial imperfections:

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$$\lambda = \varphi(\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_N), \tag{1}$$

where N is the number of terms considered in the expansions. We also assume that we are given the function φ, as well as the joint probability density

$$f_{\overline{X}}(\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_N) = \cdots \mathrm{Prob}\left[\bigcap_{i=1}^{N}(\overline{\xi}_i \leqslant \overline{X}_i \leqslant \overline{\xi}_i + \Delta\overline{\xi}_i)\right] \tag{2}$$

of the random initial imperfection vector, denoted by $X = (\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_N)^{\mathrm{T}}$, i.e., the probability that the random components $\overline{X}_i$ of the vector $\overline{X}$ will belong to the interval $(\overline{\xi}_i, \overline{\xi}_i + \Delta\overline{\xi}_i)$, where $\Delta\overline{\xi}_i$ is an increment. Due to the assumed randomness of the initial imperfection, the associated buckling load turns out to be a random variable also, denoted by Λ. Note that the random variables are denoted by capitals, whereas the possible values they can take on are denoted by lowercase notations. Bolotin (op. cit.) applied this method to a cylindrical panel, under uniform compressive load along its curved edges, with the initial imperfections represented by a single normally distributed amplitude parameter. A single-term Bubnov–Galerkin approximation yielded an equation of type (1). Conceptually, such one-term analysis is quite straightforward. Once a relation of type (1) is obtained, and the probability density of the initial imperfection $\overline{X}_{i_0}$ specified or assumed (i_0 is the index of the governing initial imperfection parameter), one can calculate the reliability of the structure. The reliability at a preselected load level α is defined as the probability that the structure will not buckle prior to α or in other words, will live beyond the "age" of α:

$$R(\alpha) = \mathrm{Prob}(\Lambda \geqslant \alpha). \tag{3}$$

Having determined the reliability of the structure, one proceeds with its design as follows: one should specify a codified reliability r, i.e., the level of reliability below which the performance of the structure is declared unacceptable. The probabilistic design criterion demands

$$R(\alpha) \geqslant r. \tag{4}$$

Inequality (4) makes it possible to solve some basic problems of stochastic buckling. If the left- and the right-hand sides of Eq. (4) are known, then one can check if the probabilistic design criterion (4) is met or violated. If some probabilistic characteristic of the initial imperfection, say its variance $d_{i_0}^2$, is unspecified, one can calculate its maximum admissible level $\max d_{i_0}$, such that the design criterion is satisfied. The value $\max d_{i_0}$ is obtained by solving the equation,

$$R(\alpha) = \mathrm{Prob}(\Lambda \geqslant \alpha | \max d_{i_0}) = r. \tag{5}$$

Solution of these types of problems may then be introduced in the quality control process. If the variance of the initial imperfection exceeds $\max d_{i_0}$, the structure is declared unacceptable. The third problem deals with the determination of the design load α_r such that if $\alpha \leqslant \alpha_r$ then the reliability will not be less than r.

The reliability of a symmetrically behaving structure at the non-dimensional load level α can be rewritten as

$$R(\alpha) = \mathrm{Prob}\big[-\overline{\xi}_1(\alpha) \leqslant \overline{X}_{i_0} \leqslant \overline{\xi}_1(\alpha)\big], \tag{6}$$

where $\overline{\xi}_1(\alpha)$ is the value at which the limit load equals α. This implies that $\overline{\xi}_1(\alpha)$ satisfies the equation,

$$\lambda = \varphi(\overline{\xi}_1) = \alpha. \tag{7}$$

Hence,

$$\overline{\xi}_1(\alpha) = \varphi^{-1}(\alpha). \tag{8}$$

If, for example, $\overline{X} \equiv \overline{X}_{i_0}$ is a random variable having a normal distribution with zero mean and mean-square deviation d_{i_0}

$$f_{\overline{X}}(\overline{x}) = \frac{1}{\sqrt{2\pi d_{i_0}}} \exp\left(-\frac{\overline{x}^2}{d_{i_0}^2} \right), \tag{9}$$

then the reliability becomes

$$R(\alpha) = \text{Prob}[-\varphi^{-1}(\alpha) \leqslant \overline{X} \leqslant \varphi^{-1}(\alpha)], \tag{10}$$

$$R(\alpha) = 2 \text{ erf}\left(\frac{\varphi^{-1}(\alpha)}{2d_{i_0}} \right), \quad \text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{t^2}{2} \right) \mathrm{d}t. \tag{11}$$

This enables one to find the probabilistic design load α_r such that if $\alpha = \alpha_r$, then the least reliability of the structure equals *r:*

$$\alpha_r = \varphi\left[2d_{i_0} \text{ erf}^{-1}\left(\frac{r}{2} \right) \right]. \tag{12}$$

The probability-based knockdown factor (PKF) is defined then as

$$\text{PKF} = \frac{\alpha_r}{P_{\text{cl}}}, \tag{13}$$

where P_{cl} is the classical buckling load.

In order to illustrate the stochastic imperfection sensitivity concept, let us consider a simple structure, namely, a column on a non-linear elastic foundation

$$EI \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + k_1 w - k_3 w^3 = -P \frac{\mathrm{d}^2 \overline{w}}{\mathrm{d}x^2}, \tag{14}$$

where $\overline{w}(x)$ is the initial imperfection, $w(x)$, the additional deflection, P, the axial load, k_1 and k_3, the non-linear spring coefficients of the foundation. Buckling of the perfect column on a linear foundation is a textbook problem (e.g. Timoshenko and Gere, 1961). An imperfect column on a non-linear softening elastic foundation exhibits imperfection sensitivity in that the limit load, the structure is able to support, may turn out to be far less than the buckling load of its perfect, linear counterpart. Application of the Galerkin method for a column simply supported at its ends yields the following equation, derived by Frazer (1965) in his Ph.D. dissertation, as a single-term approximation:

$$(1 - \lambda)^3 = \frac{81}{32} s \overline{\xi_m^2} \lambda^2, \tag{15}$$

where $\overline{\xi}_m$ is the initial imperfection amplitude associated with m half sine waves in the axial direction, λ is the non-dimensional limit load, and s is a value depending on the physical parameters of the system.

This type of analysis can also be demonstrated on the imperfection sensitivity of a shell with a non-axisymmetric periodic imperfection, studied by Koiter (1963a,b):

$$w_0(x) = gh\left(\cos \frac{i_c \pi x}{L} + 4 \cos \frac{i_c \pi x}{2L} \cos \frac{i_c \pi y}{2L} \right), \tag{16}$$

where x is the axial coordinate, y, the circumferential coordinate, g, the non-dimensional initial imperfection amplitude, i_c, the number of half waves at which the associated perfect shell buckles, L, the shell length, and h, the shell thickness. Koiter arrived at the following equation relating the buckling load with the initial imperfection amplitude:

$$(1 - \lambda)^2 + 6cg\lambda = 0, \tag{17}$$

where $\lambda = P_{\text{lim}}/P_{\text{c}}$, with P_{lim} being the limit load, P_{c}, the classical buckling load of the perfect shell, $c = \sqrt{3(1 - \nu^2)}$.

The analysis based on a single-term approximation is quite similar in its general form, but not in its particulars, to the asymptotic method developed by Koiter (1945, 1963, 1967, 1970) and by Budiansky and Hutchinson (1964). The asymptotic expressions or the equations based on a single-term Bubnov–Galerkin approximation can be utilized for understanding the disastrous influence of initial imperfections on the load-carrying capacity of the structure.

As far as the probabilistic considerations are concerned, we are looking for a highly reliable performance, associated with the probability of failure, say 10^{-4} or even less. Realizing this, one immediately should cast a doubt on the possibility that the highly simplified expressions (which, however, are of *extreme* importance in capturing the physical phenomenon itself) would accurately predict the required high reliability. In other words, simplified expressions may be unusable in calculating extremely small probabilities of failure. But this is exactly what the society demands – namely small, if not negligible, unreliability. This was also an essence of Koiter's (1980) Communication.

It was perfectly valid to utilize a single-term Bubnov–Galerkin approximation in the early work of Bolotin (1962). Analogously, application of Koiter's asymptotic expressions by Thompson (1967), Roorda (1972), and Hansen and Roorda (1973) served the purpose of illustrating the reliability approach in imperfection sensitive structures (see also Augusti, 1974). Yet, it appears that an industrial design firm, for example, cannot use such expressions in order to justify the reliability calculations with the required extremely small probabilities of failure; inclusion of additional terms in the Bubnov–Galerkin or asymptotic expansions may significantly alter the resulting probabilities of failure, and invalidate the designs proposed on the basis of single-term or asymptotic approximations. Yet, some very recent works still utilize the deterministic asymptotic expansions for reliability calculations.

We must assume that these fine points were perfectly understood by some investigators quite early, as they did not follow the deterministic single-term or asymptotic methodologies, in conjunction with treating the imperfection amplitude as a random variable.

2. Studies based on ergodicity assumption

In his review paper, Amazigo (1976) stresses with regard to Eq. (1):

> "*It is however a non-trivial problem to obtain Eq. (1) and perform the analysis for $N>2$, say. It is this difficulty that limits the effectiveness of this method.*"

Instead of utilizing the concept of random variable, as in the works by Bolotin (1962) and Thompson (1967), the scholars of the Harvard group correctly decided to adopt the theory of random functions, identifying the initial imperfections as random fields with specified probabilistic characteristics, namely, the mean initial imperfection function and the autocorrelation function. Apparently, such studies dealing with imperfection sensitive structures were first undertaken by Frazer (1965) and Frazer and Budiansky (1969). They studied the imperfect column on a non-linear elastic foundation. The length of the column was taken to be infinity. The following assumptions were made about the initial imperfection field: (a) they were considered to form a weakly homogeneous random field, (b) the assumption of ergodicity of this field was also introduced.

Weak homogeneity implies that the mean initial imperfection function is a constant, whereas the autocorrelation function depends only on the difference, $x_2 - x_1$, where x_1 and x_2 are the spatial coordinates. Hence, the mean square value of a homogeneous random function is a constant too. Weak homogeneity, or insensitivity to a shift of the initial cross-section of reference, is possible for infinite domains. Therefore, possibly, the infinite length assumption was adopted. For solving the problem, Frazer and Budiansky (1969) resorted to the classical method of stochastic linearization and to the additional assumption, that the

output random field, namely the additional deflection of the column, was ergodic too. The main conclusion derived in this article is that each infinite column in the ensemble has the *same*, *deterministic buckling load*, which depends on the autocorrelation function of the initial imperfection *alone* and not on a particular realization of any of them.

It appears to us that the source of the above paradoxical result stems from the fact that the authors assumed ergodicity not only of the *input* field, but also of the *output* field. This assumption allowed us to facilitate the analytical solutions that were derived. In order to check the validity of such an assumption, Scheurkogel et al. (1981) undertook an investigation of a model system, which allowed us to obtain a closed form solution. Then the same problem was solved by invoking the ergodicity assumption. A control parameter k was introduced so that one could study the changing behavior of the system as the control parameter was varied. It turned out that, in general, the output of the system was *inergodic*. At some value of the parameter, namely $k=2$, the ergodicity assumption yielded a result coinciding exactly with the response obtained. This implies that sometimes the error may not affect the estimate of the system's response (the ergodicity assumption constituted a beneficial, "good" error!). In two distinctive ranges of variation of the parameter k, the behavior turned out to be of different nature. For $0 \leqslant k < 2$, the ergodicity assumption introduced a small error of the order of one per cent. Yet, for $2 < k \leqslant 4$, the ergodicity assumption led to large errors. In particular, when k tends to 4, the ergodicity-based solution is finite, whereas the exact solution is unbounded. As is seen, extreme caution must be exercised when invoking the ergodicity assumption: the differential equation itself, rather than an analyst, should be given freedom to decide if the output is ergodic or not.

3. Monte Carlo simulation is not a method of last resort

In 1969, Frazer and Budiansky authored a thought provoking article about the buckling of a column with stochastic imperfections; they concluded that the realizations of the columns were different, yet they all shared the same deterministic buckling load. This load depended on the probabilistic characteristic of the imperfection, yet it was *shared* by all other columns with a probability of unity. Interestingly, this load depended on a single value of the spectral density of the initial imperfections, and was independent, otherwise, of the spectral content of the field.

To illustrate this point, let us reproduce the formulas for *deterministic* buckling loads derived for shells. Amazigo (1969) obtained the following expression for the static buckling load of a circular cylindrical shell:

$$(1-\lambda_s)^{7/2} = \frac{9\pi c^2}{2\sqrt{2}} S(1), \tag{18}$$

where $S(1)$ is the power spectral density of the imperfection spectrum $S(\omega)$ evaluated at the frequency corresponding to the classical asymmetric buckling mode, $\omega = 1$. Later, Amazigo and Budiansky (1972) presented a modified formula

$$(1-\lambda_s)^{7/2} = \frac{9\pi c^2}{2\sqrt{2}} S(1)\lambda_s^2. \tag{19}$$

According to the authors, this formula should provide a more accurate estimate for λ_s, the buckling load of the imperfect shell. Tennyson et al. (1971) proposed to evaluate $S(1)$ by utilizing the power spectral density corresponding to an exponential cosine autocorrelation function

$$S(\omega) = \frac{\Delta^2 \zeta(\omega^2+\zeta^2+\gamma^2)}{\pi[\omega^4 + 2(\zeta^2-\gamma^2)\omega^2 + (\zeta^2+\gamma^2)^2]}, \tag{20}$$

where Δ^2 is the mean square value of the imperfection. By setting $\omega = 1$, $\zeta = 0.2$ and $\gamma = 1$, they obtained

$$S(1) = 2.52\Delta^2/\pi \tag{21}$$

with Eq. (19) taking the form

$$(1-\lambda_s)^{7/2} = \frac{9c^2}{2\sqrt{2}}(2.52\Delta^2)\lambda_s^2. \tag{22}$$

Amazigo and Budiansky (1969) cautioned that use of Eq. (19) will lead to incorrect results if the actual power spectral density does not peak near $\omega = 1$.

However, it appeared to the present writer that the very constancy of the buckling load should have been questioned. It was decided that instead of pursuing some new, purely analytic approach, in addition to what was already undertaken in several studies at Harvard University, it would be nice to perform an experiment. Yet, how would one obtain numerous realizations of real columns on non-linear elastic foundations? If not in a real laboratory, then may be in a virtual one, on the computer? Thus, the idea occurred to study the Frazer–Budiansky problem by the Monte Carlo simulation.

The idea did not seem to be fanciful or even new. Indeed, Frazer (1965) himself had already performed the Monte Carlo analysis of a column on a non-linear foundation. Yet, most unfortunately, he limited himself to a single-term approximation. Such an analysis leads, once this assumption is made, to a closed-form solution, given in Eq. (11), for the reliability. Naturally, one does not need the Monte Carlo solution if the exact solution is at hand, unless one wants to illustrate the validity of the Monte Carlo solution in the particular case capable of exact solution. Once confidence has been gained through such a comparison, if it is favorable, one resorts to the multi-mode Monte Carlo solution, where the exact solution is unavailable. The Monte Carlo method had to be combined with multi-mode approximation rather than with the single-term approximation, performed by Frazer (1965).

Multi-term Monte Carlo simulation was suggested by Hansen (1977) in his probabilistic analysis of randomly imperfect shells. However, the analysis performed could be characterized as unbalanced: detailed analytical treatment combined with simplified probabilistic analysis, the assumption being that all Fourier coefficients used in the series expansion are *identically* distributed (specifically, each Fourier coefficient was taken as a normally distributed variable with the same variance). This assumption corresponds in essence to the "white-noise" autocorrelation function of the initial imperfections. Thus, the analysis did not allow information on general autocovariance function. Indeed, there exists no compelling reason for initial imperfections to be a spatial "white noise." Some other investigators, albeit in a dynamic context, also essentially followed this analysis. They neglected the correlations between the various Fourier coefficients, but adopted non-constant variances (with a new term coined for such imperfections: "grey noise") (Lindberg, 1988).

It was realized by this writer that the multi-dimensional probability densities should serve as a point of departure for the simulation analysis; the correlation analysis start from the mean function and the autocorrelation function, and end with the variance–covariance matrix of the Fourier coefficients of the initial imperfections or of any other random variables, stemming from a suitable discretization. This matrix, in the general case, must be a fully populated one, and not a diagonal one with identical (Hansen, 1977) or different (Lindberg, 1992) elements. In order to study the Frazer–Budiansky model structure, the present writer developed a general simulation procedure for solving the stochastic boundary value problems (Elishakoff, 1979).

This simulation procedure was applied to several probabilistic problems: impact buckling of a column (Elishakoff, 1978), Hoff's problem of buckling of a column in a testing machine (Elishakoff, 1980) and the Frazer–Budiansky problem Elishakoff, 1979). In the latter article, a column of *finite* length was studied for several reasons: (1) an assumption of infinite length may simplify the analytical analysis but complicate the numerical one, (2) the structures utilized and analyzed by engineers (fortunately) do not possess infinite length, (3) the non-linear column on a non-linear elastic foundation in the previous studies was not ana-

lyzed in terms of the edge-effect method to justify looking for interior solutions (as those associated with the infinite structure), and for corrective edge-effect solutions. Moreover, why should one assume infinite length when the Monte Carlo solution for realistic finite-length shells is easier? In the study of Elishakoff (1979), for each realization, the buckling load was determined numerically by transforming the non-linear algebraic equations into numerically solved ordinary differential equations.

1. The Monte Carlo solution yields results that are practically coincident with the exact solution, when the latter is obtainable for the single-term approximate problem. This demonstrates that the Monte Carlo solution may exhibit a better performance than what the various statistical tests may predict.
2. A single-term Bubnov–Galerkin approximation is not sufficient for accurate prediction of the structural reliability; depending on the system's parameters, various degrees of approximation, higher than one, must be achieved for the reliability estimates to be accurate.
3. The design buckling load associated with high reliability may significantly deviate from the average buckling load.
4. As the length of the column increases, the variance of the buckling load decreases.

The last conclusion may represent a link with the result of Frazer and Budiansky (1969) who concluded that for an infinite column, the buckling load was a deterministic quantity. However, for the realistic finite column, the buckling load depends on the particular realization of the initial imperfection function, which in turn depends on the probabilistic measures (mean and autocorrelation function) of the initial imperfections.

In a latter analysis, Day (1980) showed that in some simple cases, the ergodicity assumption may be dispensed with for evaluation of the *mean buckling load*. Yet, the analyses yielding the mean buckling load alone could hardly be considered practical. Each of us may remember various, sometimes entertaining, objections to the average quantities, and they will not be recapitulated here. Anyway, knowledge of the average buckling load is insufficient for probabilistic design of structures subject to buckling.

The development of a general simulation procedure for initial imperfections with given mean and autocorrelation functions ascertained the way of introducing the initial imperfection sensitivity into design. It involves three main items:

(a) development of accurate deterministic (analytical or numerical) tools for buckling load prediction;
(b) compilation of extensive experimental information on imperfections, boundary conditions, elastic properties, load scatter etc., with a view to deriving the mean functions and autocorrelation functions of random fields, and assessing their distributions; and
(c) utilization of the Monte Carlo analysis through simulation of brothers and sisters (not perfect clones!) of the experimentally measured structures.

Clearly, the future of shell buckling analysis is identified by Arbocz in his recent articles with the notion of stochasticity (see e.g. Arbocz, 1997).

In accordance with the recommendation of Professor Koiter (1982), the early study (Elishakoff, 1979) was generalized to include both the quadratic and cubic non-linearities of the elastic foundation (Elishakoff, 1985).

4. Reliability of shells, or how beautiful theories have to accomodate ugly facts

Before proceeding further, it is advisable to review some other deterministic developments. Fortunately, some investigators of thin shells were interested in experimental analyses.

Arbocz (1968) devoted his Ph.D. dissertation to careful experimental measurements of initial imperfections of shells, designed and manufactured in the laboratory of the California Institute of Technology. Arbocz and Babcock (1969) reported their results on the buckling experiments. They measured the initial imperfections and the prebuckling behavior of the electroplated isotropic shells via automated scanning

mechanisms. Singer et al. (1971) reported the results of analogous imperfection surveys on integrally machined ring-stiffened as well as stringer-stiffened shells. These studies provided the data for possible correlation of experimental buckling results with the theoretical or numerical prediction of the buckling loads. Such a correlation study would establish feasibility of direct incorporation of the initial imperfection measurements into the (deterministic) theoretical procedure or the numerical code, with the attendant direct comparison between the results. These results were reported by Babcock (1974), Arbocz (1974) and other authors.

"Better" results were also reported in literature. Arbocz (1974, p. 236, Table 5) reports the result of Hofmeister (1972), where *both* the experimental and theoretical buckling loads of Mylar, normalized to the classical buckling load, constituted 0.52. Such a perfect coincidence is an exception rather than an already achieved goal. Usually, when such a coincidence takes place, one wonders if it was by pure chance, and if either the experiment or the theoretical evaluation were properly conducted. For example, the following question arises: were these Mylar shells analyzed as isotropic or orthotropic ones? (Singer, 1997).

Experimental verification of Koiter's special theory has been given by Tennyson and Muggeridge (1969). They tested a series of photoelastic plastic circular shells containing an axisymmetric imperfection. The specimens were manufactured by a special spin-casting technique which yielded near perfect cylinders. Experimental points were all within about 10% of Koiter's special theory; remarkably, in some cases the error was of the order of 2%. For the stringer-stiffened shell AS-2, Arbocz (1981) reported a value (produced by computer code STAGS via 30 node model) of 243.8 N/cm, whereas the experimental value 226.3 N/cm was about 7% off. For shell A8, Arbocz reported a non-dimensional experimental buckling load 0.66, whereas the theoretical prediction was 0.69. For the shell designated as AB-6, the experimental non-dimensional buckling load was 0.75, whereas the theoretical estimate was 0.72 (Singer, 1983, Table 27.5). In parallel, Makarov (1969) carried out extensive measurements of the initial imperfection profiles at the Moscow Power Engineering Institute with a view of studying their statistical characteristics.

Encouraged by the mere possibility of a relatively good correlation between the experimental and numerical results (and hopefully even a better one, once the boundary conditions could be *closely* identified), it occurred to the present author that the probabilistic analysis must be linked with the numerical and experimental developments in a hybrid manner. The experimental developments at Caltech, the Technion, the University of Toronto, and the Moscow Power Engineering Institute led this writer to the idea of trying, in parallel, to investigate the effects of the ergodicity assumption suggested at Harvard.

The first step was to probabilistically evaluate the initial data banks just compiled (Arbocz and Abramovich, 1979; Arbocz, 1982). Two group of shells, designated as A-shells and B-shells respectively were statistically analyzed (Elishakoff and Arbocz, 1982a,b) from the compiled data bank. Estimated variances of the measured initial imperfections were plotted as functions of the axial coordinate for the A group (Fig. 7 in Elishakoff and Arbocz, 1982a,b) and B group (ibid, Fig. 10). These variances were *not constant*, implying that the experimental initial imperfections cannot be treated as a weakly homogeneous random field, let alone as an ergodic one. This demonstrated that the works based on the ergodicity hypothesis could not be characterized as practical. The fact that in another collection of shells, measured at the Moscow Power Engineering Institute, the experimental data did not contradict the assumption of weak homogeneity of the initial imperfections of circular cylindrical shells in the circumferential direction (Makarov, 1969, 1970) appears to be surprising, as the latter shells had a seam.

Neither did the experimental results support the assumption made by Hansen (1977), namely, that the Fourier coefficients of the initial imperfections were statistically independent and identically distributed, nor did the imperfections constitute a "grey" noise as suggested by Lindberg (1988). Thus, the statistical analysis of real shells vividly illustrated that *none* of the specialized assumptions made in literature about the probabilistic pattern of the initial imperfections were justified. It should be immediately noted, out of fairness to the authors of the above studies, that these assumptions were not made arbitrarily: they either permitted theoretical treatments of various kinds (the work performed at Harvard university), or reduced

the computational effort (the work performed at Moscow Power Engineering Institute or the University of Toronto).

Thus, the assumption of homogeneity in the axial direction, adopted in the West, as well as its circumferential counterpart adopted in the East, did not prove to be viable hypotheses. In addition, dealing with infinitely long shells in conjunction with the hypothesis of ergodicity diverted the attention of the investigators from the main aspect of reliability of the structure, which can be utilized for design purposes. On the contrary, single-term Bubnov–Galerkin approximations, while concentrating on the reliability, could not reflect the reality properly, as too much was neglected.

Indeed, recourse to the *finite* structure combined with the ability to determine *both* the mean values and variances of the buckling loads in the multi-modal setting was a major step, but the article by Miller and Hedgepeth (1979) did not address the main issue of reliability determination; reliability, rather than the first and second moments, can be utilized in design. One may argue that the Gaussian assumption for the buckling loads would yield the reliability once the first and second moments were calculated. Such an assumption would be incorrect, for the probability density is highly skewed (see e.g. Elishakoff, 1983, Figs. 5.27 and 11.9) as both the single- or multi-mode analyses would clearly demonstrate.

A special simulation procedure (Elishakoff, 1978) was applied to shells with axisymmetric imperfections (Elishakoff and Arbocz, 1982a,b) as well as to shells with general non-axisymmetric imperfections. The assumption of uncorrelatedness of some of the Fourier coefficients, adopted in the latter article, was subsequently dispensed with (Elishakoff, 1988).

The next step in the analysis, following the Monte Carlo simulation of the "brothers" and "sisters" of experimentally measured shells, is the performance of the buckling calculations of each simulated shell by the special procedures. These procedures include a special theory by Koiter (1963a, b) for asymmetric shells, the multi-mode analysis of Arbocz and Babcock (1976), the finite element method (Ernst, 1979) for the general non-axisymmetric shells, as well as other analytical studies or numerical codes, like STAGS, etc.

Once a large amount of realization was available, the reliability of the shells could be computed as the fraction of shells that did not fail prior to a predetermined load level. This "*assume as little as possible*" approach in dealing with the problem, directly combined three major ingredients, hitherto unconnected: the theoretical, numerical and experimental aspects of the buckling research.

5. How to corroborate the Monte Carlo analysis?

The Monte Carlo method is usually resorted to when an exact solution is not available. However, some other approximate method may be available to tackle the stochastic boundary value problems in question and once such an approximate solution has been obtained, the investigators almost invariably check it against the Monte Carlo simulation. Both methods are approximate in their nature; which is preferable in these circumstances? Many investigators think that the non-Monte Carlo method is preferable as a "cheaper" technique, but this premise is due for reexamination.

Two considerations are of importance in this respect. The Monte Carlo simulation technique is a universal tool, applicable for example, for small or large deviations of the random variables or functions involved. The other approximate techniques invariably have limited areas of application. They may be effective, for example, where the coefficients of variations are small. When we perform both the Monte Carlo analysis and the other approximate evaluation, which method *checks* the other then?

Usually, researchers "promote" their own approximate technique; then, as they maintain, they compare it with the Monte Carlo method, as the exact solution is unavailable. Thus, they check a method with limited range of application with the one with a wider validity. Still, these authors could claim that they are comparing a numerically cheap technique with a numerically expensive method, and if the comparison turns out to be good, they advocate for use of the non-Monte-Carlo method. Then it appears reasonable to

claim that in the general case of the complex structure, or when stochastic variations are not small, the Monte Carlo method should be used as a universal technique. Yet, as it is, in essence, a computerized experimental method, the other, approximate analytical techniques should be used to check it. Such a conclusion may appear paradoxic at first sight, but whereas for a specific set of parameters, one approximate method may be valid and the other may prove to be applicable for other sets. Both should be in the vicinity of the results furnished by the Monte Carlo method, albeit in different ranges. It becomes obvious that the general Monte Carlo method is being checked by methods of limited applicability. An analogous view is shared by Shinozuka (1996).

It appears imperative to evaluate the results obtained by the Monte Carlo method by different analytical–numerical techniques which may be effective in specific ranges of variation of the parameters. One such technique is the second-order second-moment method. For its detailed exposition, the reader may consult the article by Hasofer and Lind (1974), and numerous later texts. This method has been extended by Elishakoff et al. (1987) to cover the non-linear buckling of shells with initial imperfections treated as random functions.

The cornerstone of the method is the availability of a deterministic state equation

$$Z = Z(X_1, X_2, \ldots, X_N), \tag{23}$$

where $Z(\ldots)$ is a *performance function*. Its nature depends on the type of the structure and the limit state considered. According to the definition, the equation

$$Z = 0 \tag{24}$$

determines the failure boundary. The inequality

$$Z < 0 \tag{25}$$

implies failure, whereas its opposite

$$Z > 0 \tag{26}$$

indicates a successful performance. The *zero-order* second-moment method calls for linearization of the function Z at the *mean* points $E(X_i)$ and knowledge of the distribution function of the random vector $\boldsymbol{X}$. Calculations are relatively straightforward if $\boldsymbol{X}$ is normally distributed, if not, an appropriate transformation is in order.

In the case under investigation, we are interested in knowing the reliability of the structure at any given load λ, i.e.

$$R(\lambda) = \mathrm{Prob}(\Lambda \geqslant \lambda), \tag{27}$$

where Λ is the random buckling load. A function Z can then be defined as follows:

$$Z(\lambda) = \Lambda - \lambda = \varphi(X_1, X_2, \ldots, X_N) - \lambda \tag{28}$$

where λ is the applied deterministic load, $\varphi(X_1, X_2, \ldots, X_N)$ is the relation postulated by Bolotin (1962) as per Eq. (1). However, when such a relationship is not available analytically, we can visualize that availability of a numerical code is *equivalent* to the knowledge of this function. To combine the numerical codes developed, for example, by Arbocz and Babcock (1980) with the zero-order or first-order second-moment method, we need to know the lower order probabilistic characteristics of Z. In the first approximation, for small variances and covariances of X_j, we have

$$\begin{aligned} E(Z) &= E(\Lambda) - \lambda = E[\varphi(X_1, X_2, \ldots, X_N)] - \lambda \\ &\simeq \varphi[E(X_1), E(X_2), \ldots, E(X_N)] - \lambda. \end{aligned} \tag{29}$$

The variance of Z is given as

$$\mathrm{var}(Z) = \mathrm{var}(\Lambda) \simeq \sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial\varphi}{\partial X_j}\right)_0\left(\frac{\partial\varphi}{\partial X_k}\right)_0 \mathrm{cov}(X_j, X_k), \tag{30}$$

where $\mathrm{cov}(X_j, X_k)$ is the covariance of components X_j and X_k of the initial imperfections, determined by experimental measurements. Calculation of the derivatives $(\partial\varphi/\partial X_j)$ at mean values of the arguments $X_{j0} = E(X_j)$ is carried out numerically. Having estimated $E(Z)$ and $\mathrm{var}(Z)$, one obtains the estimate for the probability of failure at the load level λ:

$$P_f(\lambda) = \mathrm{Prob}(Z < 0) = \Phi(-\beta); \quad \Phi(x) = \tfrac{1}{2} + \mathrm{erf}(x), \tag{31}$$

where

$$\beta = E(Z)/\sigma_Z. \tag{32}$$

σ_Z being the mean square deviation of Z and

$$\sigma_Z = \sqrt{\mathrm{var}(Z)}. \tag{33}$$

Numerical analysis for unstiffened circular cylindrical shells were reported by Elishakoff et al. (1987), and showed good correlation with the Monte Carlo method. A more accurate method is the first-order second-moment method or (as it is universally referred to) the *Level 2* method. In the buckling context, it is implemented as follows (Elishakoff, 1984): for simplicity, we consider a problem involving two initial imperfection parameters X_j and X_k, forming a random vector $\boldsymbol{X}$. Through the initial imperfection data banks one obtains the mean values $E(X_j), E(X_k)$ and the variance–covariance matrix

$$[C] = \begin{bmatrix} \mathrm{var}(X_j) & \mathrm{cov}(X_j, X_k) \\ \mathrm{cov}(X_j, X_k) & \mathrm{var}(X_k) \end{bmatrix}. \tag{34}$$

We denote the vector of *basic variables* as $\boldsymbol{Y}$, they have zero means and unity variances. Then

$$\boldsymbol{X} = C^{1/2}\boldsymbol{Y} + E(\boldsymbol{X}), \tag{35}$$

where

$$\boldsymbol{Y} = [C^{1/2}]^{-1}[\boldsymbol{X} - E(\boldsymbol{X})], \tag{36}$$

and $C^{1/2}$ is the square root of a positive-definite matrix C. For every realization of Y, we find the realization of X immediately, and using the computer programs developed by Arbocz and Babcock (1980), we determine the buckling load. This enables us to determine the failure boundary for fixed values of α

$$Z = \alpha - \lambda = 0 \tag{37}$$

as follows: one specifies the direction

$$l = (i\ \cos\gamma + j\ \sin\gamma)l. \tag{38}$$

At the top of the vector, we check if the boundary load exceeds α. If it does not, we lengthen the vector and repeat the procedure, and if the buckling load still does not exceed α, we continue the process. If the buckling load is in excess of α, we multiply the length of the vector by a number less than unity and repeat the process till we reach the point at which the buckling load is nearly α, within the demanded accuracy. Then we rotate the vector l by changing the angle γ, and repeat the process. This yields the failure boundary, and the smallest distance to it (provided there are no multiple points with the same minimum distance) is determined from the origin of coordinates. This distance is denoted by β_{HL}, as a Hasofer-Lind index. The probability of failure is given by Eq. (30), where β is replaced by β_{HL}. This procedure (Elishakoff, 1984) awaits its numerical implementation for buckling of structures.

The zero-order second-moment method has been applied by Arbocz and Hol (1991) and Arbocz (1997) for integrally stringer-stiffened shells (see Stam, 1996). Computationally, it is much less expensive than the direct Monte Carlo method, but the latter is applicable for *arbitrary* coefficients of variations, whereas the former is valid only for *small* variations of the initial imperfection amplitudes. This basic premise may not hold true for many cases reported in the initial imperfection data banks. In addition, the reliability estimates furnished by the zero-order and first-order second-moment reliability estimates may be quite substantial.

It appears that the methods developed in the past two decades in collaboration with Professor Arbocz illustrate that the *imperfection* sensitivity concept can be introduced into practice. This will enable the theoretical findings to be directly embodied in codes, instead of well known "knockdown factors."

It should be stressed that we are not trying to "match" the existing knockdown factors; if this were the case, the probabilistic methods would not have a predictive power and only constitute "forecasting the past." Probabilistic methods produce not arbitrary knockdown factors, but the ones that are directly derivable from the required reliability levels. This may eliminate both underdesign and overdesign of structures, prone to buckling. However, we must emphasize that extremely intelligent efforts are needed for probabilistic methods to be implemented in practice. Sophisticated devices are needed to measure initial imperfection profiles. Fortunately, this is possible, as was demonstrated at various laboratories in different parts of the world and especially by Professor Arbocz at the Delft University of Technology. The results of these measurements have to be statistically interpreted, with a view to checking hypotheses on their distribution. If Fourier coefficients are involved, *joint* probability distributions are needed rather than *marginal* ones. The probabilistic analysis must be coupled with accurate deterministic analysis, based on either FEM or multi-mode Galerkin approximations. If the mean square deviations of the initial imperfections are small compared with their mean values, one can use the zero-order or the first-order second-moment method; if the coefficient of variation is moderate or large, it appears that one should use the Monte Carlo method. In direct realization of this technique without variance reduction techniques, one may need a supercomputer environment or a massively parallel computing facility for a sufficient amount of Monte Carlo simulations so as to determine sufficiently small probabilities of failure with acceptable accuracy.

Here let us backtrack for a moment: have we not neglected some important facet of the problem? For example, what about the boundary conditions? This problem is discussed in Section 6.

6. Correct modeling of boundary conditions is an extremely non-trivial task

To be able to determine small probabilities of failure, one needs extremely accurate deterministic theories. Buckling load calculations naturally involve compliance with the boundary conditions. How can they be modeled with sufficient accuracy? For beams, determination of the boundary conditions appears to be a treatable task. Studies by Horton et al. (1969) and Sweet et al. (1976, 1977) may support such a conclusion. For shells, however, their determination appears to be an extremely complex task. Existing studies predict relatively small influence of the boundary conditions for isotropic unstiffened shells (Almroth, 1966), if displacement of the edges in the circumferential direction is restrained. However, their influence is significant for stiffened shells, as was demonstrated by Arbocz and Sechler (1976). Buckling loads for the specific integrally stiffened shell XS-1 with the SS-3 and SS-4 boundary conditions, respectively were 141.6 and 184.5 lb/in. Yet, for the same shell with the C-3 and C-4 boundary conditions, the buckling loads were respectively 161.6 and 204.0 lb/in. Experimental data on stiffened shell buckling were reported by Singer et al. (1971).

Elishakoff et al. (1992) used the following boundary conditions:

$$\alpha N_x + \beta u = v = w = M_x = 0, \tag{39}$$

where u and v are the displacements in axial and circumferential directions, respectively, w is the transverse displacement, N_x is the axial force per unit length, M_x is the bending moment per unit length; α and β are the coefficients describing elastic springs at the boundaries.

These generalized boundary conditions are reducible to those given in Eqs. (39) and (40). Specifically, for $\alpha = 0$ and non-zero β, the boundary condition reduces to SS-4 boundary conditions, whereas for $\beta = 0$ and non-zero α we recover SS-3 conditions. For non-zero α, Eq. (39) is rewritten as

$$N_x + K_x u = 0, \quad K_x = \beta/\alpha. \tag{40}$$

For optimal resemblance to the experimental realization of the boundary conditions, K_x in Eq. (40) should be treated as a function of the circumferential coordinate θ. The following dependence was chosen

$$K_x(\theta) = \begin{cases} K_1, & \text{for} \quad \theta_2 i \leqslant \theta \leqslant \theta_{2i+1}, \\ K_2, & \text{for} \quad \theta_{wi+1} \leqslant \theta \leqslant \theta_{2i+2}, \end{cases} \tag{41}$$

where the serial number i varies from zero to eight, $\theta_0 = 0°$, $\theta_{18} = 360°$. In addition, $K_2 \ll K_1$. This implies that the shell is attached to the apparatus with nine relatively strong springs with stiffness K_1 and nine relatively weak springs with stiffness K_2.

It should be stressed that uniform axial springs have been considered by Singer (1962) in the shell buckling, in a linear setting. Non-uniform boundary conditions along the circumference, in the context of boundary imperfections, were first treated by Hoff and Soong (1967) with identical non-uniformity at both edges of the shell; if this non-uniformity can be characterized as a boundary imperfection, one can say that they considered "perfect imperfections." This assumption has been abandoned by Stavsky et al. (1988) and Sabag et al. (1989) who considered the realistic "imperfect imperfections," i.e. non-identical imperfections at both the edges.

Although the problem of non-uniform support conditions for non-linear, imperfect shells was formulated by Elishakoff et al. (1992), numerical results have yet to be reported. This latter article led to better appreciation of the difficulties in modeling "true" boundary conditions. Recently, another close look at the realization of boundary conditions in experimental setting was undertaken (Arbocz, 1997).

Once this is done, nagging questions (by the pseudo-skeptics in ourselves) will still remain. Here is their partial list:

(a) Is there a scatter in the values of the axial spring K_x?

(b) If the answer is yes, then how to model K_x as a random variable or as a one-dimensional random field?

(c) If K_1 and K_2 can be treated as random variables, how to determine experimentally their probabilistic characteristics, e.g. the mean values $E(K_1), E(K_2)$, the variances $\mathrm{var}(K_1), \mathrm{var}(K_2)$ and even more important, the covariance $\mathrm{cov}(K_1, K_2)$?

(d) Moreover, how to find their joint probability distribution for rigorous probabilistic analysis?

(e) How to predict, on-line, the boundary conditions in the service environment, when connections between the different parts of the system are possibly changed or damaged?

Probabilistic modeling does not appear to be the most suitable answer to the last question. It was shown by Elishakoff and Fang (1995) (see also Ben-Haim and Natke, 1992; Ben-Haim, 1996) that non-probabilistic convex modeling may turn out to be suitable for partial answers on the partial information available. In particular, one can attempt to determine convex sets to which the spring constants belong, rather than their exact deterministic values, or their probabilistic characteristics.

A trivial conclusion of this section is that rigorous modeling of boundary conditions is a highly non-trivial task. A less transparent conclusion is the recognition of the fact that special *identification* techniques are needed for direct incorporation of the boundary conditions in the analysis. For example, in the deterministic setting, it is important to closely approximate the axial spring coefficient K_x as a function of the circumferential coordinate; in the probabilistic setting, one needs to identify the probabilistic characteristics

of the random field $K_x(\theta)$. In convex modeling, one only has to determine the set to which K_x belongs. Effective deterministic codes are needed to incorporate the variation of $K_x(\theta)$ in the numerical analysis. The problem is not resolved even then: in other shells, with different sets of unknown boundary conditions, additional spring constants may be needed, to accurately represent realistic boundary conditions, enormously complicating the analysis.

7. Probability is not a magic wand

As we see, in the probabilistic setting, the imperfection sensitivity concept may appear to be nice and dandy. Yet, do probabilistic methods solve the problem in its entirety? Do probabilistic methods have disadvantages, or do they constitute a panacea for fully closing the chasm between the theory and practice?

For a partial reply to this question, let us recapitulate that once the probability of failure is determined, we must design the ensemble of shells. This is done by requiring that the reliability should not be less than some preselected value r. Some investigators adopt, without providing a justification, a value of allowable probability of failure $P_{\mathrm{f,all}} = 1 - r = 0.02$ $(r = 0.98)$. Yet, it is not easy to convince the top management that, following the frequency interpretation of probability, nearly two in every 100 realizations of the structure may fail.

Initial imperfection data banks, even when compiled, may still contain *insufficient* information for rigorous probabilistic processing of all variables. In such circumstances, researchers "randomize" the problem by assigning the probability distributions. By doing so they try to "make something out of nothing" and create the illusion of availability of information, while in actuality it is lacking. Is such a procedure a necessary evil, and should one just live with it? At least, many investigations felt uncomfortable with this situation.

One of the architects of the probabilistic applications to mechanics, Freudenthal (1956) stressed that "... ignorance of the cause of variation does not make such variation random." (for the detailed discussion of this and other related topics see Elishakoff, 1999a)

Although Freudenthal recognized, as this quotation may demonstrate, that probability must not be the only game in town, he did not provide any alternative to it. Yet, this noble self-criticism of his "own" methods, appears worthy of following.

This and numerous other considerations led Ben-Haim (1985, 1996), Ben-Haim and Elishakoff (1989, 1990), Elishakoff et al. (1994) to develop the method of *convex modeling* for applied mechanics applications. The name stems from realization of the fact that most inequalities describing the range of variations of uncertain variable constitute convex sets. As the variables are defined by their ranges of variation only rather than by the probability densities, the following questions can be posed:

(a) What is the maximum buckling load the structure may experience when the initial imperfections vary in a convex set?

(b) What is the minimum buckling load the structure may attain in these circumstances?

Once these questions are answered, it is prudent to use the minimum buckling load as design load.

Such analyses were performed by Ben-Haim and Elishakoff (1990), Ben-Haim (1993a,b, 1994), Lindberg (1992), Pantelides (1996a,b), Elishakoff et al. (1994), and Elseifi et al. (1999). The initial imperfection vector $\boldsymbol{X}$ was represented as the sum of a nominal vector $\boldsymbol{X}_0$ and the deviation vector $\boldsymbol{\zeta}$. The deviation was postulated to fall within the following ellipsoidal set:

$$Z(\chi, \omega) = \left\{ \boldsymbol{\zeta} : \sum_{i=1}^{N} \frac{\zeta_i}{\omega_i^2} \leqslant \chi^2 \right\}, \tag{42}$$

where the size parameter χ and the semi-axes $\omega_1, \omega_2, \ldots, \omega_N$ are based on experimental data, obtainable from the initial imperfection data banks. The lowest buckling load which can be obtained for any of the

shells in the ensemble described by Eq. (42) is expressed formally as the minimum of expression (1) on the set Z:

$$\mu(\chi,\omega)=\min_{\zeta\in Z(\chi,\omega)}\varphi(\boldsymbol{X}_0+\boldsymbol{\zeta}). \tag{43}$$

Hence $\mu(\chi,\omega)$ is the buckling load of the "weakest" shell in the ensemble Z, constructed to represent a realistic range of shells. The limit load for an imperfection vector $\boldsymbol{X}_0+\boldsymbol{\zeta}$, to the first order of $\boldsymbol{\zeta}$, is

$$\varphi(\boldsymbol{X}_0+\boldsymbol{\zeta})=\varphi(\boldsymbol{X}_0)+\sum_{i=1}^{N}\frac{\partial\varphi(X_0)}{\partial X_i}\zeta_i. \tag{44}$$

Thus the problem (43) is replaced by the following

$$\mu(\chi,\omega)=\min_{\zeta\in Z(\chi,\omega)}\left[\varphi(\boldsymbol{X}_0)+\phi^{\mathrm{T}}\boldsymbol{\zeta}\right], \tag{45}$$

where

$$\phi^{\mathrm{T}}=\left(\frac{\partial\varphi(X_0)}{\partial X_1},\frac{\partial\varphi(X_0)}{\partial X_2},\ldots,\frac{\partial\varphi(X_0)}{\partial X_N}\right), \tag{46}$$

where the superscript T stands for matrix transposition. The minimum buckling load is given by the formula (Ben-Haim and Elishakoff, 1989):

$$\mu(\chi,\omega)=\varphi(X_0)-\chi\left[\sum_{i=1}^{N}\left(\omega_i\frac{\partial\varphi(X_0)}{\partial X_i}\right)^2\right]^{1/2}. \tag{47}$$

From this relation, one recognizes that significant reduction of the buckling load results from high sensitivity of the nominal buckling load to Fourier coefficients, whose semi-axes in the imperfection ellipsoid are large. We also recognize that the minimum buckling load depends linearly on the overall size χ of the imperfection ellipsoid, and non-linearly on its shape parameters $\omega_1,\omega_2,\ldots,\omega_N$ and on the partial derivatives $\partial\varphi(X_0)/\partial X_i$. The values of the partial derivatives have been borrowed from a previous probabilistic study by Elishakoff et al. (1987).

Whereas the formula (49) is a first-order approximation, a second-order approximation has also been written explicitly in terms of the Hessian matrix with elements $\partial^2\varphi(x_0)/\partial\zeta_i\partial\zeta_j$. The result is not recapitulated here.

It is also of interest to define the variations of the imperfections in terms of a radial tolerance on the shape of the shell. Manipulations which are not reproduced here lead to the following expression of the buckling load in terms of the imperfection deviation

$$\varphi(\boldsymbol{X}_0+\boldsymbol{\zeta}(\xi,\theta))=\varphi(\boldsymbol{X}_0)+\int_0^{2\pi}\int_0^{\pi}\zeta(\xi,\theta)S(\xi,\theta)\,\mathrm{d}\xi\,\mathrm{d}\theta, \tag{48}$$

where $S(\xi,\theta)$ is a combination of trigonometric functions with coefficients that depend on the elements of the vector $\partial\psi(\boldsymbol{X}_0)/\partial\zeta_i$; for details one may consult Ben-Haim and Elishakoff (1989, 1990). A close examination of Eq. (48) reveals that the largest reduction in the buckling load is obtained from the imperfection profile which switches between its extreme values $\hat{\zeta}$ and $-\hat{\zeta}$, where $\hat{\zeta}$ is the radial tolerance. The minimum buckling load for the ensemble of shells with radial tolerance $\hat{\zeta}$ reads:

$$\begin{aligned}\mu(\hat{\zeta})&=\min_{|\zeta|<\hat{\zeta}}[\varphi(\boldsymbol{X}_0+\boldsymbol{\zeta}(\xi,\theta))]\\&=\varphi(\boldsymbol{X}_0)-\hat{\zeta}\int_0^{2\pi}\int_0^{\pi}|S(\xi,\theta)|\mathrm{d}\xi\mathrm{d}\theta.\end{aligned} \tag{49}$$

Suppose now that one wishes to construct a radial tolerance for which the minimum buckling load takes on the value λ_0. Then one chooses $\hat{\zeta}$ as follows

$$\hat{\zeta} = [\mu(\hat{\zeta}) - \varphi(\boldsymbol{X}_0)] \Big/ \int_0^{2\pi} \int_0^{\pi} |S(\xi, \theta)| \mathrm{d}\xi \mathrm{d}\theta. \tag{50}$$

This approach permits theoretical determination of the knockdown factor within the convex modeling (CKF). It is defined as the ratio of the minimum buckling load to the classical buckling load:

$$\mathrm{CKF} = \frac{1}{P_{\mathrm{cl}}} \left\{ \varphi(\boldsymbol{X}_0) - \chi \left[\left(\sum_{i=1}^{N} \omega_i \frac{\partial \varphi(X_0)}{\partial X_i} \right)^2 \right]^{1/2} \right\} \tag{51}$$

for ellipsoidally by modeled initial imperfections, and

$$\mathrm{CKF} = \frac{1}{P_{\mathrm{cl}}} \left\{ \varphi(\boldsymbol{X}_0) - \hat{\zeta} \int_0^{2\pi} \int_0^{\pi} |S(\xi, \theta)| \mathrm{d}\xi \, \mathrm{d}\theta \right\} \tag{52}$$

for shells with radial tolerance (compare Eq. (13), defining the KF within the probabilistic modeling). This knockdown factor is anticipated to lie *above* those provided by the NASA monographs (1969), which would imply that the existing monographs specify too conservative estimates and thus *penalize* carefully designed shells. As we see, convex non-probabilistic modeling of uncertainty, provides theoretical means for determining the KF.

For other applications of this method the reader may consult the articles by Elishakoff and Ben-Haim (1990), Lindberg (1992), Ben-Haim (1993a, b), Pantelides (1996a,b), and Elseifi et al. (1999).

The "competition" between the probabilistic and convex modeling was discussed by Elishakoff et al. (1994). It was shown that in many realistic circumstances, convex modeling of uncertainty and the probabilistic analysis yield close results. This partially answers the following question: which of them is superior? Another facet of the problem is the fact that convex modeling is both conceptually *simpler* and computationally *less expensive* than probabilistic analysis. This may suggest, in accordance with the law of parsimony, or *Ockham's razor*, that it is often the preferable one (see also Elishakoff, 1999b).

8. Optimization and anti-optimization can be combined

Non-probabilistic models of uncertainty, in essence, look for the *worst* designs under uncertainty constraints. One determines the minimum buckling load that the structure may experience when uncertain parameters (initial imperfections, elastic moduli, or other properties) vary within some sets. The ranges within which the uncertainties vary are the only quantities that are known; the probability densities may be unknown due to the lack of sufficient information for an accurate probabilistic model. This situation is somewhat opposite to what we are looking for in classical optimization of structures, which looks for the *best* designs. It appeared natural, therefore, to the present writer to coin the term *anti-optimization* for such analysis under uncertainty (Bazant and Gedolin, 1991). It covers, as particular cases, convex modeling (including interval analysis or ellipsoidal modeling) as well as non-convex, set-theoretical modeling.

Optimization and anti-optimization techniques can be meaningfully combined. Indeed, one is interested in *maximizing* the *minimum* buckling load the structure can carry due to uncertainty in the system. Such analyses were carried out by Adali et al. (1994, 1997). Zhu et al. (1996) developed a novel technique incorporating experimentally available information into an ellipsoid of minimum volume, in an N-dimensional space of initial imperfections.

9. Conclusion

Along with achievements, we have highlighted some misconceptions, mistakes or misuses of probabilistic methods in buckling of structures as a learning experience. As a Chinese proverb maintains, "Through mistakes we can learn the truth." Thus, these mistakes and their critical analysis become a path which enables us to formulate questions more rigorously, and provide more insightful answers than before (as G.K. Chesterton maintained, "It isn't that they can't see the solution. It is that they can't see the problem"). Obviously, "they" must be replaced by a collective "we". An additional reason for emphasizing past mistakes is: (a) to warn against repeating them by not yet experienced researchers or engineers, (b) to allow researchers to learn from others' mistakes, rather than commit them themselves; (c) to demonstrate that even in the modern non-judgmental society, some decisions and judgments must be made without getting lost in the available literature, on the one hand, is being over-attracted by the computer's possibilities on the other. The latter is especially unfortunate, as overzealous researchers often declare all variables (except, luckily, universal constants) to be random; then, by pushing the computer button one can choose any marginal distribution desired from the list of available ones; thus, quick estimates of the probabilistic parameters are furnished. This naive and deceiptful research consumes minds and resources which could be more usefully utilized. Links with experimental data, hypotheses testing subroutines on joint probability density are not only absent, but not even dreamt about.

This article clearly shows that closing the chasm between theory and practice is not an impossible task. It can be dealt with along several alternative avenues. The probabilistic methods do not seem to have a monopoly on the truth about uncertainty. Conceptually and computationally, simpler approaches are possible, especially when data is scarce. In some instances, hybrid approaches may be of use.

The problem is not closed. Whereas the knockdown factor disregards available theoretical knowledge, its advent was an ingenious idea, as it yielded safe designs. Still, it somehow mixes "apples" and "oranges," i.e. structures produced by various manufacturing methods with different degrees of workmanship, while creating universal criteria, irrespective of the specific manufacturing process.

It is time now to *differentiate* the knockdown factors for different manufacturing methods, and formulate new guidelines for NASA and other agencies. It seems to this writer that research should concentrate on several directions:

1. Accumulation of data for statistical analysis to check the nature of the distribution of random initial imperfections, elastic moduli, thickness variations, load variations etc.
2. Development of techniques for identification of boundary conditions, which may turn out to have a non-uniform nature. When limited data is provided, identification may be replaced by establishment of local modifications of the boundary conditions during the service life of the structure, via convex modeling.
3. Development of finite element codes in a stochastic setting, incorporating uncertain imperfections, elastic moduli, boundary conditions; thickness variation, and loading conditions development of buckling pre- and post-processors for commercially available codes like NASTRAN, ADINA, ALGOR, DIANA, etc.
4. Interrelation between probabilistic methods and antioptimization in the buckling context (see Elishakoff, 1999a,b).

Numerous other topics on uncertain buckling, dealt with either in probabilistic or antioptimization settings are treated in the forthcoming monograph (Elishakoff et al., 2000).

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